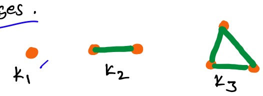
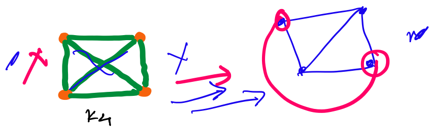
Lecture 9: Graph 3

Wednesday, April 29, 2026 10:49 PM

## IX Planar Graphs

A graph G is planar if and only if can be drawn without any crossing edges.



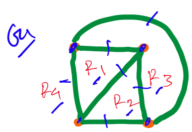




We can do multiple experiments to find out if a graph is planar. However this is not efficient.

**Euler's Theorem!**

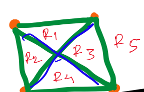
Let G be a connected planar graph with $|V| = v$ and $|E| = e$. Let $r$ be the number of regions determined by the planar embedding. Then we can conclude that

$$\boxed{v - e + r = 2}$$




$v - e + r = 2$

$4 - 6 + 4 = 2$




$\boxed{v - e + r = 2}$

$4 - 6 + 5 \neq 2$

**Caveat!**

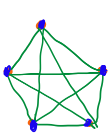
To use this formula, we need to first draw the graph and then determine, which may defeat the purpose.

However, there are corollaries that come from Euler's Theorem that can help us.

Let G be a loop-free and planar graph with $|V| = v$ and $\boxed{|E| = e > 2}$ and $r$ regions. Then,

$3r \le 2e$ (w/ region) and $e \le 3v - 6$ (w/o region) True

Example, Show that $K_5$ is non-planar



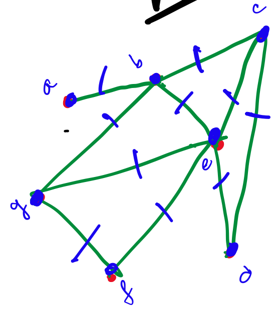
$e \le 3v - 6$

$10 \le 3(5) - 6$

$10 \le 9$

Therefore, $K_5$ is non-planar.

**H**




$v - e + r = 2$

$7 - 10 + r = 2$

$r = 12 - 7$

$= 5$

$3r \le 2e$ (w/ region)

$3(5) \le 2(10)$

$15 \le 20$

$e \le 3v - 6$ (w/o region)

$10 \le 3(7) - 6$

$10 \le 21 - 6$

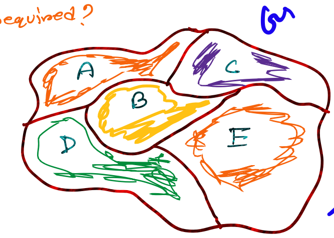
$10 \le 15$

## Graph Coloring

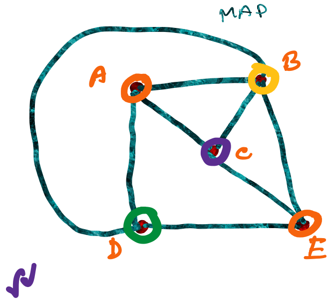
Problem! A continent is divied into several ... countries do not get countries. Neighboring countries [illegible] along and therefore do not want the same color on their flag.

The flag makers, trying to reduce costs, wants to use as few colors as possible.

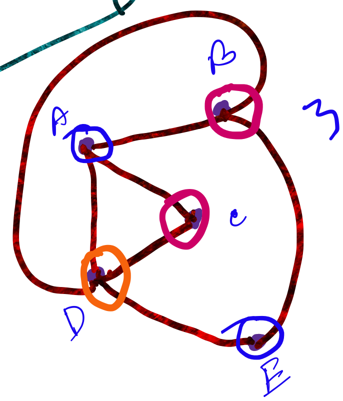
what is the minimum number of colors required?





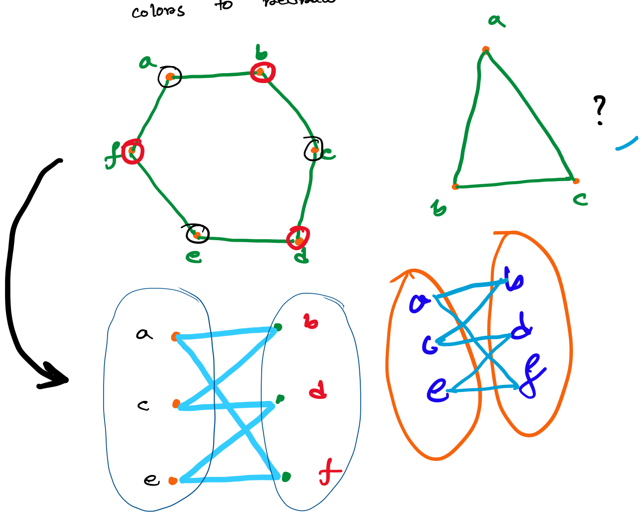



Dual Graph




Graph Coloring — Bipartite Graphs

we talked about graph coloring briefly in our last class. we could use colors to redraw a graph as bipartite.




⇒ Chromatic Number – complete graphs

we call the minimum number of colors required to color the vertices of a graph the chromatic number, $\chi(G)$.

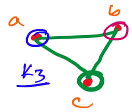
Therefore, $\chi(G)$ for any bipartite graph is 2

Let's find out $\chi(G)$ for complete graphs $K_2$, $K_3$, and $K_4$, and see if we can generalize for $K_n$.

$K_2$

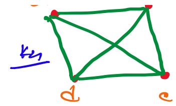

$\chi(G) = 2$

$K_3$




$\chi(G) = 3$

$K_4$




$\chi(G) = ?$

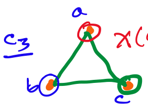
what can we say for $K_n$?

$\chi(K_n) = n$

⇒ Chromatic Number - Cycle graph

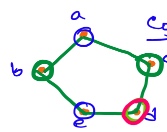
Find $\chi(G)$ for cycle graphs $C_3, C_4, C_5$, and $C_6$ below and then generalize for $C_n$.

$C_3$



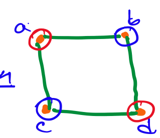

$\chi(C_3) = 3$

$C_5$



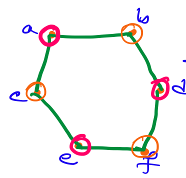

$\chi(C_5) = 3$

$C_4$




$\chi(C_4) = 2$

$C_6$




$\chi(C_6) = 2$

$\chi(C_n) = 2$ if $n$ is odd

$\chi(C_n) = 3$ if $n$ is even